Final Exam — Analysis (WBMA012-05)

Monday 30 January 2023, 18.15h–20.15h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering “yes”, “no”, or “42” is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (5 + 10 = 15 points)

Consider the following set:

$$A = \left\{ \frac{2x+1}{x+1} \,:\, x > 0 \right\}.$$

(a) Show that $u = 2$ is an upper bound for A.

(b) Prove that $\sup A = 2$.

Problem 2 (10 + 5 = 15 points)

Consider the following sequence:

$$x_{n+1} = \sqrt{1 + x_n} \quad \text{with} \quad x_1 = 1.$$

(a) Apply the Monotone Convergence Theorem to prove that (x_n) is convergent.

(b) Show that $\lim x_n = (1 + \sqrt{5})/2$.

Problem 3 (5 + 5 + 5 = 15 points)

Consider the set $A = \{x \in \mathbb{Q} \,:\, 0 < x < 1\} = \mathbb{Q} \cap (0,1)$.

(a) Is A open?

(b) Is A closed?

(c) Is A compact?

Please turn over for problems 4, 5 and 6!

Problem 4 (15 points)

It is given that $f(x) = \ln(x)$ is differentiable on $(0, \infty)$. Apply the Mean Value Theorem to prove that f is uniformly continuous on $A = [a, \infty)$ where $a > 0$.

Problem 5 (5 + 10 = 15 points)

Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq 1/n \\ 1 & \text{if } x > 1/n. \end{cases}$$

(a) Determine a function $f : [0, 1] \to \mathbb{R}$ such that $f_n \to f$ pointwise.

(b) Is the convergence uniform on $[0, 1]$?

Problem 6 (15 points)

Consider the function $f : [0, 2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}$$

For $0 < \epsilon < 4$ consider the following partition of the interval $[0, 2]$:

$$P = \{0, 1 - \epsilon/4, 1 + \epsilon/4, 2\}.$$

Compute the lower sum $L(f, P)$ and the upper sum $U(f, P)$, and show that f is integrable.

End of test (90 points)

Solution of problem 1 (5 + 10 = 15 points)

(a) For any $x > 0$ we have the following inequality

$$\frac{2x+1}{x+1} < \frac{2x+2}{x+1} = \frac{2(x+1)}{x+1} = 2.$$

So for every $a \in A$ we have shown that $a < 2$, which implies that $u = 2$ is an upper bound for A.
(5 points)

(b) *Method 1.* Let u be any upper bound for A. Then for every $n \in \mathbb{N}$ we have

$$\frac{2n+1}{n+1} \leq u.$$

(2 points)

By the Algebraic Limit Theorem it follows that

$$\lim \frac{2n+1}{n+1} = \lim \frac{2+1/n}{1+1/n} = \frac{\lim(2+1/n)}{\lim(1+1/n)} = 2.$$

(3 points)

By the Order Limit Theorem it follows that $2 \leq u$.
(3 points)

So we have shown that any upper bound u of A satisfies $2 \leq u$. By the definition of supremum it follows that $\sup A = 2$.
(2 points)

Method 2. Let $\epsilon > 0$ be arbitrary. For $x > 0$ we have the following equivalent statements:

$$\begin{aligned} 2-\epsilon < \frac{2x+1}{x+1} \quad &\Leftrightarrow \quad (2-\epsilon)(x+1) < 2x+1 \\ &\Leftrightarrow \quad 2-\epsilon < \epsilon x + 1 \\ &\Leftrightarrow \quad 1-\epsilon < \epsilon x \\ &\Leftrightarrow \quad \frac{1-\epsilon}{\epsilon} < x. \end{aligned}$$

(5 points)

We conclude that for every $\epsilon > 0$ there exists an element $a \in A$ (namely, the element $a = (2x+1)/(x+1)$ with $x > (1-\epsilon)/\epsilon$) such that $2 - \epsilon < a$. This shows that any number $u < 2$ can is no longer an upper bound for A. Therefore, $\sup A = 2$ (we have used Lemma 1.3.8 here).
(5 points)

Solution of problem 2 (10 + 5 = 15 points)

(a) We first show that $x_n < x_{n+1}$ for all $n \in \mathbb{N}$. Since $x_1 = 1$ and $x_2 = \sqrt{2}$ this is certainly true for $n = 1$. Now assume that for some $n \in \mathbb{N}$ we have $x_n < x_{n+1}$. This gives $x_n + 1 < x_{n+1} + 1$. Since the square root is an increasing function we have

$$\sqrt{1 + x_n} < \sqrt{1 + x_{n+1}},$$

which precisely means that $x_{n+1} < x_{n+2}$. By induction our claim has been proven.
(4 points)

Next, we show that $x_n < 2$ for all $n \in \mathbb{N}$. Since $x_1 = 1$ this is certainly true for $n = 1$. Now assume that for some $n \in \mathbb{N}$ we have $x_n < 2$. This gives $1 + x_n < 3 < 4$. Since the square root is an increasing function we obtain

$$\sqrt{1 + x_n} < \sqrt{4} = 2,$$

which precisely means that $x_{n+1} < 2$. By induction our claim has been proven.
(4 points)

The Monotone Convergence Theorem states that an increasing sequence for which all terms are bounded from above is convergent. We conclude that the given sequence (x_n) is convergent.
(2 points)

(b) Let $x = \lim x_n$. Then $x = \lim x_{n+1}$ as well. Since the square root is a continuous function, we conclude that

$$x = \lim x_{n+1} = \lim \sqrt{1 + x_n} = \sqrt{1 + \lim x_n} = \sqrt{1 + x}.$$

(3 points)

This gives the quadratic equation $x^2 - x - 1 = 0$, which has the solutions $x = (1 \pm \sqrt{5})/2$. By the Order Limit Theorem it follows that only the positive solution can be the limit of the sequence (x_n).
(2 points)

Solution of problem 3 (5 + 5 + 5 = 15 points)

(a) The element $a = 1/2$ belongs to A. For any $\epsilon > 0$ the set $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$ contains irrational numbers. Therefore the inclusion $V_\epsilon(a) \subseteq A$ does not hold since A only contains rational numbers. We conclude that A is not open.
(5 points)

(b) The sequence (a_n) given by $a_n = 1/2n$ belongs to A. We have that $\lim a_n = 0$. If A were closed, then $0 \in A$ as well. However, we know that $0 \notin A$. We conclude that A is not closed.
(5 points)

(c) If A were compact, then A would be both closed and bounded. In part (b) we concluded that A is not closed. Therefore, A is not compact.
(5 points)

Solution of problem 4 (15 points)

Assume that $a > 0$, and let $x, y \in [a, \infty)$ with $x \neq y$. By the Mean Value Theorem there exists a point c between x and y such that

$$\ln(x) - \ln(y) = \frac{x - y}{c}.$$

(3 points)

Taking absolute values and noting that $c > a$ it follows that

$$|\ln(x) - \ln(y)| < \frac{1}{a}|x - y|$$

(3 points)

Let $\epsilon > 0$ be arbitrary and take $\delta \leq a\epsilon$. Then for any $x, y \in [a, \infty)$ we have the following implication:

$$|x - y| < \delta \quad \Rightarrow \quad |\ln(x) - \ln(y)| < \frac{\delta}{a} \leq \frac{a\epsilon}{a} = \epsilon.$$

This shows that the function $f(x) = \ln(x)$ is uniformly continuous on the set $[a, \infty)$.
(9 points)

Solution of problem 5 (5 + 10 = 15 points)

Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \le x \le 1/n \\ 1 & \text{if } x > 1/n. \end{cases}$$

(a) Fix any number $x \in [0,1]$. If $x = 0$, then $f_n(x) = 0$ for all $n \in \mathbb{N}$ and thus $\lim f_n(x) = 0$. If $x > 0$, then for all natural numbers $n > 1/x$ we have that $1/n < x$ which implies that $f_n(x) = 1$ and thus $\lim f_n(x) = 1$. We conclude that $f_n \to f$ pointwise, where

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x \le 1. \end{cases}$$

(5 points)

(b) *Method 1.* All functions f_n are continuous. Indeed, each f_n is trivially continuous on the intervals $[0, 1/n)$ and $(1/n, 1]$. For fixed n we have

$$\lim_{x \to 1/n} f_n(x) = 1 = f_n(1).$$

(5 points)

If the convergence $f_n \to f$ were uniform on $[0,1]$, then f would be continuous on $[0,1]$ as well. In particular, f would be continuous at $x = 0$. Since f is not continuous at $x = 0$, we conclude that the convergence cannot be uniform.
(5 points)

Method 2. For all $n \in \mathbb{N}$ we have that

$$f_n(x) - f(x) = \begin{cases} 0 & \text{if } x = 0, \\ nx - 1 & \text{if } 0 < x \le 1/n, \\ 0 & \text{if } x > 1/n. \end{cases}$$

(2 points)

We claim that

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup\{1 - nx \,:\, x \in (0, 1/n]\} = 1.$$

Indeed, for $x \in (0, 1/n]$ we have $1 - nx < 1$, so $u = 1$ is an upper bound for the set in the right hand side. For any $\epsilon > 0$ taking $x < \epsilon/n$ gives $1 - \epsilon < 1 - nx$, so any number $u < 1$ cannot be an upper bound. This proves the claim.
(5 points)

Finally, we conclude that the statement

$$\lim \left(\sup_{x \in [0,1]} |f_n(x) - f(x)| \right) = 0$$

does *not* hold. By a theorem proved in the lectures it follows that the convergence $f_n \to f$ is not uniform on $[0,1]$.
(3 points)

Solution of problem 6 (15 points)

We have $n = 3$ subintervals of $[0, 2]$ defined by the following points:

$$x_0 = 0, \quad x_1 = 1 - \epsilon/4, \quad x_2 = 1 + \epsilon/4, \quad x_3 = 2.$$

Over the subintervals we have the following suprema:

$$\begin{aligned} M_1 &= \sup\{f(x) \,:\, x \in [x_0, x_1]\} = 1, \\ M_2 &= \sup\{f(x) \,:\, x \in [x_1, x_2]\} = 1, \\ M_3 &= \sup\{f(x) \,:\, x \in [x_2, x_3]\} = 1. \end{aligned}$$

This gives the following value for the upper sum:

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n}(x_k - x_{k-1}) = x_n - x_0 = 2.$$

(6 points)

Over the subintervals we have the following infima:

$$\begin{aligned} m_1 &= \inf\{f(x) \,:\, x \in [x_0, x_1]\} = 1, \\ m_2 &= \inf\{f(x) \,:\, x \in [x_1, x_2]\} = 0, \\ m_3 &= \inf\{f(x) \,:\, x \in [x_2, x_3]\} = 1. \end{aligned}$$

This gives the following value for the upper sum:

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = (x_1 - x_0) + (x_3 - x_2) = 2 - \epsilon/2$$

(6 points)

With the partition P we have

$$U(f, P) - L(f, P) = \epsilon/2 < \epsilon.$$

This shows that f is integrable.
(3 points)

Remark: in all fairness, we have only shown the existence of a partition P for which $U(f, P) - L(f, P) < \epsilon$ when $0 < \epsilon < 4$ and not for *all* $\epsilon > 0$. However, the case $\epsilon \geq 4$ is easily tackled. Just take $P = \{0, 1\}$ for which we have $U(f, P) - L(f, P) = 2 < \epsilon$.
(not taken into account in grading)